TEMPERATURE FIELDS AND DISPLACEMENT OF PLATES OF LINEARLY VARIABLE THICKNESS

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The problem of thermoelastic bending of plates of linearly variable thickness is solved by the method of a small parameter.

Let a plate of the linearly variable thickness $h(r) = h_0[1+\lambda(2\frac{r}{a}-1)]$ bounded by a plane on one side and by a conical surface of large apex angle at the other (Fig. 1) be deformed under the action of a uniformly distributed load of intensity q_z and a temperature field with given temperatures T^*_1 and T^*_2 of the surfaces bounding the plate. It is assumed that the plate is simply supported along the outline and its endfaces are heat insulated.

1. The problem of determining the temperature field in a plate reduces to the solution of the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{T}}{\partial r}\right)+\frac{\partial^2 \hat{T}}{\partial z^2}=0 \tag{1}$$

under the following conditions:

$$\begin{gathered}\hat{T}(0,\ z)<\infty;\ \hat{T}'_r(a,\ z)=0;\\ \hat{T}_{\mathbf{low.b.}}=T_1^*;\ \hat{T}_{\mathbf{upp.b.}}=T_2^*.\end{gathered} \tag{2}$$

We introduce the small parameter λ in (1) by using the change of variables

$$\rho=\frac{r}{a},\ \xi=\frac{z+\frac{h_0}{2}(1-\lambda)\cos\varphi_0}{h\cos^3\varphi_0/\cos 2\varphi_0}, \tag{*}$$

where φ_0 is the angle between the normal to the middle surface and its axis. For $[\lambda(h_0/a)]^2 \ll 1$ we have the representation

$$\varphi_0=\operatorname{arctg}\left(\lambda\frac{h_0}{a}\right)=\sum_{k=0}^{\infty}(-1)^k\frac{\left(\lambda\frac{h_0}{a}\right)^{2k+1}}{2k+1}.$$

We seek the desired solution of (1) in the form

$$T(\rho,\ \xi)=\sum_{k=0}^{\infty}\lambda^k T_k(\rho,\ \xi), \tag{3}$$

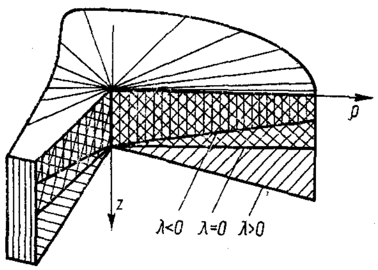

Fig. 1. Profile of a plate as a function of the sign of λ.

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where $T(\rho, \xi)$ is the transformed function $\hat{T}(r, z)$ upon substitution of (*). The first two coefficients of the series (3) are found from the equations

$$\frac{\partial^2 T_0}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial T_0}{\partial \rho} + \frac{1}{t^2} \frac{\partial^2 T_0}{\partial \xi^2} = 0, \tag{4}$$

$$\frac{\partial^2 T_1}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial T_1}{\partial \rho} + \frac{1}{t^2} \frac{\partial^2 T_1}{\partial \xi^2} = -2(2\rho - 1) \frac{\partial^2 T_0}{\partial \rho^2} + $$
$$+ 4\xi \frac{\partial^2 T_0}{\partial \rho \partial \xi} - \frac{2(2\rho - 1)}{\rho} \frac{\partial T_0}{\partial \rho} + \frac{2\xi}{\rho} \frac{\partial T_0}{\partial \xi}, \tag{5}$$

where $t = h_0/a$. The boundary conditions

$$T_k(\rho, 0) = T_1^*;\ T_k(\rho, 1) = T_2^*;\ T_k(0, \xi) < \infty;\ (T_k)'_\rho (1, \xi) = 0,\ k = 0, 1 \tag{6}$$

are appended to (4) and (5).

The following expression can be obtained for $T_0(\rho, \xi)$ by the Fourier method:

$$T_0(\rho, \xi) = T_1^* + (T_2^* - T_1^*)\xi. \tag{7}$$

We seek the solution of (5) in the form of the series

$$T_1(\rho, \xi) = \sum_{k=1}^{\infty} u_k(\xi) J_0(\mu_k \rho), \tag{8}$$

where $J_0(\mu_k \rho)$ is the zeroth-order Bessel function of the first kind. Substituting (8) into (5), we find an equation for u_k after manipulation

$$\frac{1}{t^2} \frac{\partial^2 u_k}{\partial \xi^2} - \mu_k^2 u_k(\xi) = 2(T_2^* - T_1^*)\xi A_k.$$

Integrating this latter with the boundary conditions taken into account, we obtain for $T_1 \cdot (\rho, \xi)$

$$T_1(\rho, \xi) = \frac{T_2^* - T_1^*}{3} t^2 A_1 (\xi^3 - \xi) + \sum_{k=2}^{\infty} \frac{(T_2^* - T_1^*) 2A_k}{\mu_k^2} \left[\frac{\operatorname{sh}(\mu_k t \xi)}{\operatorname{sh}(\mu_k t)} - \xi \right] J_0(\mu_k \rho). \tag{9}$$

Limiting ourselves to two terms of the series in (8), we arrive at a two-term asymptotic formula for the temperature distribution in the plate

$$T(\rho, \xi) = T_1^* + \Delta T^* \xi + \lambda \left\{ \frac{\Delta T^*}{3} A_1 t^2 (\xi^3 - \xi) + \sum_{k=2}^{\infty} \frac{2\Delta T^* A_k}{\mu_k^2} \left[\frac{\operatorname{sh}(\mu_k t \xi)}{\operatorname{sh}(\mu_k t)} - \xi \right] J_0(\mu_k \rho) \right\}, \tag{10}$$

where $\Delta T^* = T^*_2 - T^*_1$, $A_k = [2 - \pi H_1(\mu_k)]/J_0^2(\mu_k)$.

2. The equilibrium equations for plates of the shape under consideration have the form [1]

$$\frac{d}{dr}(N_r r) - N_\theta + \frac{d}{dr}(\varphi_0 r Q) + q_r r = 0, \tag{11}$$

$$\frac{d}{dr}(Qr) - \frac{d}{dr}(\varphi_0 r N_r) + q_z r = 0, \tag{12}$$

$$\frac{d}{dr}(M_r r) - M_\theta - Qr = 0, \tag{13}$$

where the notation in [1] has been used.† For the generalized forces and moments used here, the following representation is valid [2]:

†In [2, p. 34], the term $\frac{d}{dr}(Qr\varphi_0)$ is neglected in the first equilibrium equation without a basis for the error induced here.

$$N_r = A\left[\frac{du}{dr} + \nu\left(\frac{u}{r} + \varphi_0\frac{w}{r}\right) - (1+\nu)\varepsilon_T\right],$$

$$N_\theta = A\left[\frac{u}{r} + \nu\frac{du}{dr} + \varphi_0\frac{w}{r} - (1+\nu)\varepsilon_T\right],$$

$$M_r = -D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} - (1+\nu)\varkappa_T\right],$$

$$M_\theta = -D\left[\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - (1+\nu)\varkappa_T\right],$$

where u, w are displacement vector components along the r and z axes; $A = Eh/(1-\nu^2)$, $D = Eh^3/12(1-\nu^2)$; E and ν are the elastic modulus and Poisson ratio; and ε_T and κ_T are generalized pure thermal strains.

The constitutive system of equilibrium equations in displacements has the form

$$\begin{gathered}(1+\varphi_0^2)\tilde{h}\rho\frac{d^2\tilde{u}}{d\rho^2} + \left\{(1+\varphi_0^2)\frac{d(\tilde{h}\rho)}{d\rho} + \nu\varphi_0^2\tilde{h}\right\}\frac{d\tilde{u}}{d\rho} + \\ + \left\{(1+\varphi_0^2)\nu\tilde{h} - \frac{\tilde{h}}{\rho}\right\}\tilde{u} + (1+\varphi_0^2)\varphi_0\nu\tilde{h}\frac{d\tilde{w}}{d\rho} + \\ + \left[(1+\varphi_0^2)\nu\varphi_0\frac{d\tilde{h}}{d\rho} - \varphi_0\frac{\tilde{h}}{\rho}\right]\tilde{w} - (1+\varphi_0^2)(1+\nu)\tilde{h}\rho\frac{d\tilde{\varepsilon}_T}{d\rho} - \\ - (1+\nu)\left[(1+\varphi_0^2)\frac{d\tilde{h}}{d\rho}\rho + \varphi_0^2\tilde{h}\right]\tilde{\varepsilon}_T + \frac{(q_r - \varphi_0 q_z)}{A_1^*} = 0,\end{gathered} \tag{14}$$

$$\begin{gathered}\tilde{h}^3\rho\frac{d^3\tilde{w}}{d\rho^3} + \left[3\tilde{h}^2\frac{d\tilde{h}}{d\rho}\rho + \tilde{h}^3\right]\frac{d^2\tilde{w}}{d\rho^2} + \left[3\nu\tilde{h}^2\frac{d\tilde{h}}{d\rho} - \frac{\tilde{h}^3}{\rho}\right]\frac{d\tilde{w}}{d\rho} + \\ + 12\nu\varphi_0^2\tilde{h}\tilde{w} + 12\varphi_0\rho\tilde{h}\frac{d\tilde{u}}{d\rho} + 12\nu\varphi_0\tilde{h}\tilde{u} - (1+\nu)\tilde{h}^3\rho\frac{d\tilde{\varkappa}_T}{d\rho} - \\ - 3(1+\nu)\tilde{h}^2\frac{d\tilde{h}}{d\rho}\rho\tilde{\varkappa}_T - 12(1+\nu)\varphi_0\rho\tilde{h}\tilde{\varepsilon}_T - \frac{12}{A_1^*}\int_0^\rho q_z\rho d\rho = 0,\end{gathered} \tag{15}$$

where $\tilde{u} = u/a$, $\tilde{w} = w/a$, $\rho = r/a$, $\tilde{\varepsilon}_T$, $\tilde{\kappa}_T = a\kappa_T$ are dimensionless variables.

The boundary conditions

$$M_r|_{\rho=1} = 0;\ N_r|_{\rho=1} = 0;\ \tilde{w}|_{\rho=1} = 0;\ \frac{d\tilde{w}}{d\rho}|_{\rho=0} = O(1) \tag{16}$$

correspond to the simple support case. The thermal strains ε_T and κ_T are determined from the formulas

$$\varepsilon_T = \frac{1}{\tilde{h}}\int_{-\frac{\tilde{h}}{2}}^{\frac{\tilde{h}}{2}} \alpha_T T d\Theta, \tag{17}$$

$$\varkappa_T = \frac{12}{\tilde{h}^3}\int_{-\frac{\tilde{h}}{2}}^{\frac{\tilde{h}}{2}} \alpha_T T\Theta d\Theta, \tag{18}$$

where α_T is the heat-conduction coefficient, and $T(\rho, \xi)$ is the temperature distribution within the plate.

Let us represent $\tilde{u}$ and $\tilde{w}$ in the form

$$\tilde{u} = \sum_{k=0}^{\infty}\lambda^k\tilde{u}_k, \tag{19}$$

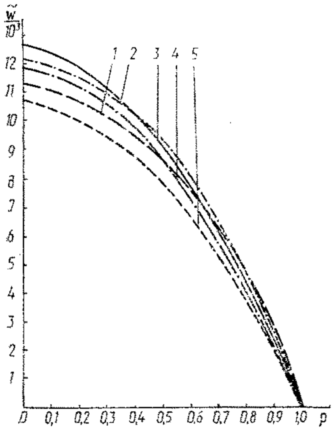

Fig. 2. Distribution of $\tilde{w}$ along the radius of a plate for $E = 2 \cdot 10^6$, $\alpha_T = 13 \cdot 10^{-6}$, $\nu = 0.25$, $t = 0.09$, $q_z = 1$, $T^*_1 = 25°C$, $T^*_2 = 190°C$: 1) $\lambda = -0.4$; 2) $\lambda = -0.2$; 3) 0; 4) 0.2; 5) 0.4.

$$\tilde{w} = \sum_{k=0}^{\infty} \lambda^k \tilde{w}_k. \tag{20}$$

Substituting (19) and (20) into (14) and (15) with (17), (18), and (10) taken into account, we obtain

$$\rho^2 \tilde{u}_0'' + \rho \tilde{u}_0' - \tilde{u}_0 = 0, \tag{21}$$

$$\rho^3 \tilde{w}_0''' + \rho^2 \tilde{w}_0'' - \rho \tilde{w}_0' = \frac{6}{A_1^* t^3} q_z \rho^4, \tag{22}$$

$$\begin{gathered}\rho^2 \tilde{u}_1'' + \rho \tilde{u}_1' - \tilde{u}_1 = (\rho^2 - 2\rho^3) \tilde{u}_0'' + (\rho - 4\rho^2) \tilde{u}_0' + \\ + [2(1-\nu)\rho - 1] \tilde{u}_0 - (2\rho - 1)[\rho^2 \tilde{u}_0'' + \rho \tilde{u}_0' - \tilde{u}_0] - \\ - \frac{12}{t} (\rho^2 \tilde{w}_0' + \nu \rho \tilde{w}_0) + \frac{q_z \rho^2}{A_1^*} + (1-\nu) 2\rho^2 \alpha_T (T_1^* - \Delta T^*) - \\ - 2\alpha_T (1+\nu) \Delta T^* \rho^2 \sum_{k=2}^{\infty} \frac{A_k}{\mu_k} J_1(\mu_k \rho) \left[\frac{1 - \mathrm{ch}(\mu_k t)}{\mu_k t \, \mathrm{sh}(\mu_k t)} + \frac{1}{2} \right],\end{gathered} \tag{23}$$

$$\begin{gathered}\rho^3 \tilde{w}_1''' + \rho^2 \tilde{w}_1'' - \rho \tilde{w}_1' = -3\rho^3 (2\rho - 1) \tilde{w}_0''' - \rho^2 (12\rho - 3) \tilde{w}_0'' - \\ - \rho^2 \left[6(\nu - 1) + \frac{3}{\rho} \right] \tilde{w}_0' + 3(1+\nu) \rho^3 \left\{ 24\alpha_T \left(\frac{\Delta T^*}{3} - \frac{T_1^*}{2} \right) + \right. \\ \left. + 12\alpha_T \frac{1}{t} \left(T_1^* - \frac{\Delta T^*}{2} \right) \right\} + \frac{(1+\nu)}{t} \rho^3 \left\{ -24\alpha_T \Delta T^* \sum_{k=2}^{\infty} \frac{A_k}{\mu_k} J_1(\mu_k \rho) \times \right. \\ \times \left[\frac{1}{\mu_k^3 t^3 \, \mathrm{sh}(\mu_k t)} - \frac{1}{3} \right] - 12\alpha_T \left(T_1^* - \frac{\Delta T^*}{2} \right) + \\ \left. + 6\alpha_T \left[-2\Delta T^* \sum_{k=2}^{\infty} \frac{A_k}{\mu_k} J_1(\mu_k \rho) \left(\frac{1 - \mathrm{ch}(\mu_k t)}{t \mu_k \, \mathrm{sh}(\mu_k t)} + \frac{1}{2} \right) \right] \right\} + \\ + \frac{12(1+\nu)}{t} \rho^3 (T_1^* - \Delta T^*) + \frac{12}{t} (\rho^3 \tilde{u}_0' + \nu \rho^2 \tilde{u}_0).\end{gathered} \tag{24}$$

The boundary conditions (16) are transformed into the form

$$\begin{gathered}\tilde{u}_0'(1) + \nu \tilde{u}_0(1) = (1-\nu) \alpha_T (T_1^* - \Delta T^*), \\ \tilde{w}_0''(1) + \nu \tilde{w}_0'(1) = \frac{1+\nu}{t} \alpha_T \Delta T^*, \\ \tilde{w}_0(1) = 0, \; \left. \frac{d\tilde{w}_0}{d\rho} \right|_{\rho=0} = O(1),\end{gathered} \tag{25}$$

$$\tilde{u}_1'(1) + \nu \tilde{u}_1(1) = (1-\nu) \alpha_T \left[A_1 t^2 \frac{\Delta T^*}{12} + \sum_{k=2}^{\infty} \frac{\Delta T^* 2A_k}{\mu_k^2} J_0(\mu_k) \left(\frac{1 - \mathrm{ch}(\mu_k t)}{\mu_k t \, \mathrm{sh}(\mu_k t)} + \frac{1}{2} \right) \right] - \nu t \tilde{w}_0(1),$$

$$\tilde{w}_1''(1) + \nu\tilde{w}_1'(1) = \frac{1+\nu}{t}\alpha_T\Delta T^*\left[12\sum_{k=2}^{\infty}\frac{A_k}{\mu_k^2}J_0(\mu_k)\left(\frac{\mu_k^2t^2(1-\mathrm{ch}(\mu_k t))+2}{\mu_k^3t^3\,\mathrm{sh}(\mu_k t)} - \frac{1}{6}\right) - \frac{t^2}{30}A_1 - 1\right],$$

$$\tilde{w}_1(1) = 0,\ \left.\frac{d\tilde{w}_1}{d\rho}\right|_{\rho=0} \quad O(1). \tag{26}$$

Integrating (21)-(24) with (25) and (26) taken into account, we obtain expressions for $\tilde{u}_0$, $\tilde{w}_0$, $\tilde{w}_1$:

$$\tilde{u}_0 = \alpha_T t\,\frac{1-\nu}{1+\nu}(T_1^* - \Delta T^*)\rho,$$

$$\tilde{w}_0 = 0{,}1875\frac{q_z}{A_1^*t^3}\left[2\,\frac{3+\nu}{1+\nu}(1-\rho^2) + \rho^4 - 1\right] - \frac{\alpha_T}{2t}\Delta T^*(1-\rho^2),$$

$$\tilde{w}_1 = \left[-\frac{2+\nu}{2}(1-\rho^2) - \frac{1+\nu}{3}(1-\rho^3)\right]\left\{1{,}5\frac{q_z}{A_1^*t^3}\,\frac{3+\nu}{1+\nu} + \right.$$

$$+ 4\alpha_T\left[2\Delta T^* - 3T_1^* - \frac{1}{t}\left(\frac{5}{2}\Delta T^* - 3T_1^* - \frac{1-\nu}{1+\nu}(T_1^* - \Delta T^*)\right]\right\} +$$

$$+ \frac{q_z}{A_1^*t^3}\left\{\frac{1-\rho^2}{1+\nu}(\nu^2 + 16{,}125\nu + 43{,}375) + 0{,}06\rho^4(\rho + 9{,}375) - 0{,}6225\right\} -$$

$$- 12\alpha_T\Delta T^*\frac{1+\nu}{t}\sum_{k=2}^{\infty}\sum_{n=0}^{\infty}\frac{A_k}{\mu_k}\,\frac{(-1)^n\left(\frac{\mu_k}{2}\right)^{2n+1}}{n!\,(n+1)!\,[(2n+4)^2 - 2(2n+4)]} \times$$

$$\times\left[\frac{1-\rho^2}{2(1+\nu)}(2n+3+\nu) + \frac{\rho^{2n+4}-1}{2n+4}\right]\left[\frac{2+\mu_k^2t^2[1-\mathrm{ch}(\mu_k t)]}{\mu_k^3t^3\,\mathrm{sh}(\mu_k t)} - \frac{1}{6}\right] +$$

$$+ \frac{\alpha_T}{2t}\Delta T^*(1-\rho^2)\left\{1 + \frac{t^2}{30}A_1 - 12\sum_{k=2}^{\infty}\frac{A_k}{\mu_k^2}J_0(\mu_k)\left[\frac{2+\mu_k^2t^2[1-\mathrm{ch}(\mu_k t)]}{\mu_k^3t^3\,\mathrm{sh}(\mu_k t)} - \frac{1}{6}\right]\right\}.$$

Numerical computations by using (19) and (20) show (Fig. 2) that taking account of the initial middle-surface curvature can result in significant corrections to the classical formulas for determining w, $M_r(\theta)$, $M_r(\theta)$, which are obtained under the assumption that the plate middle surface is a plane ($\lambda = \varphi_0 = 0$).

NOTATION

h(r), plate thickness; a, outer radius; λ, small parameter characterizing the change in plate thickness along the radius; $A_1^* = E/(1 - \nu^2)$; μ_k, roots of the first-order Bessel function of the first kind J_1; N, Q, M_r, normal force, transverse force, and bending moment acting in a section normal to the middle surface; N_θ, M_θ, normal force and bending moment acting in a section tangent to the middle surface; H_1, Struve function.

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